CS 2650: Paper Discussion

FLASHATTENTION: Fast and Memory-Efficient Exact Attention with IO-Awareness

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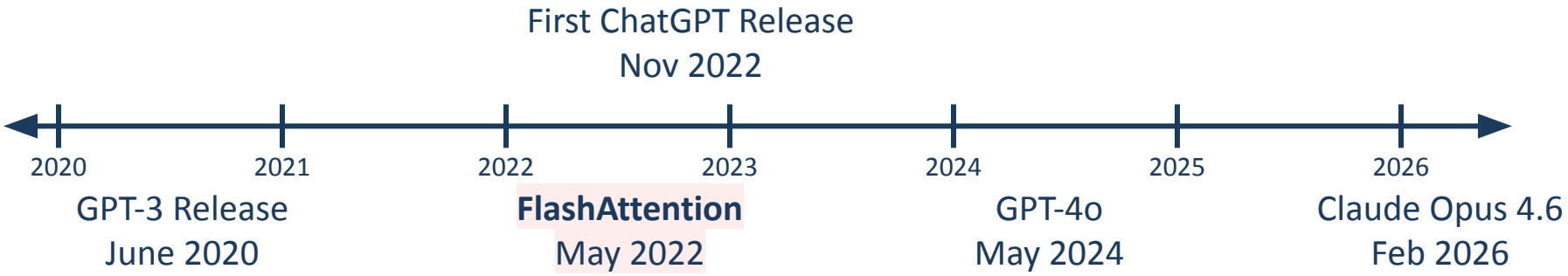
AI
LLM
Transformers
Attention

Timeline



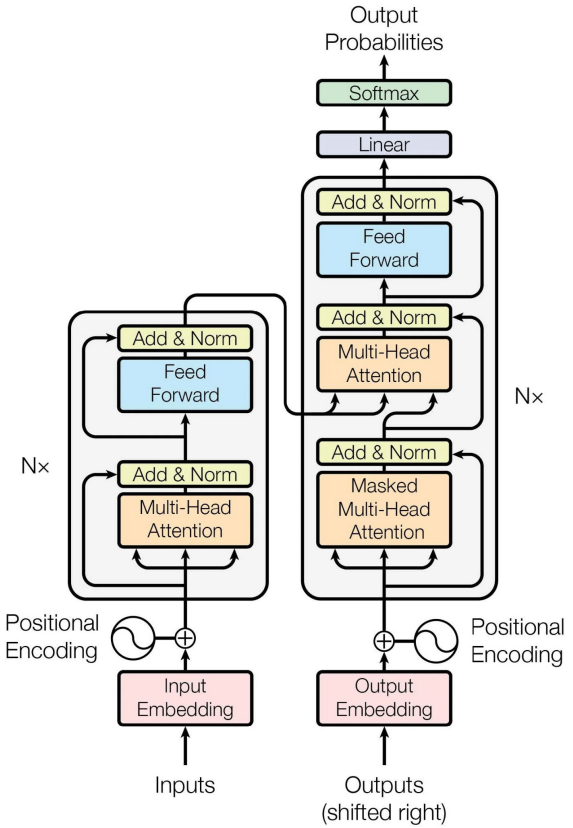
This paper was written before LLMs were mainstream

Introduction

Transformers Are Everywhere

Longer context enables:

- Long-document reasoning
- Code understanding
- Retrieval augmentation
- Genomics



Attention Is All You Need

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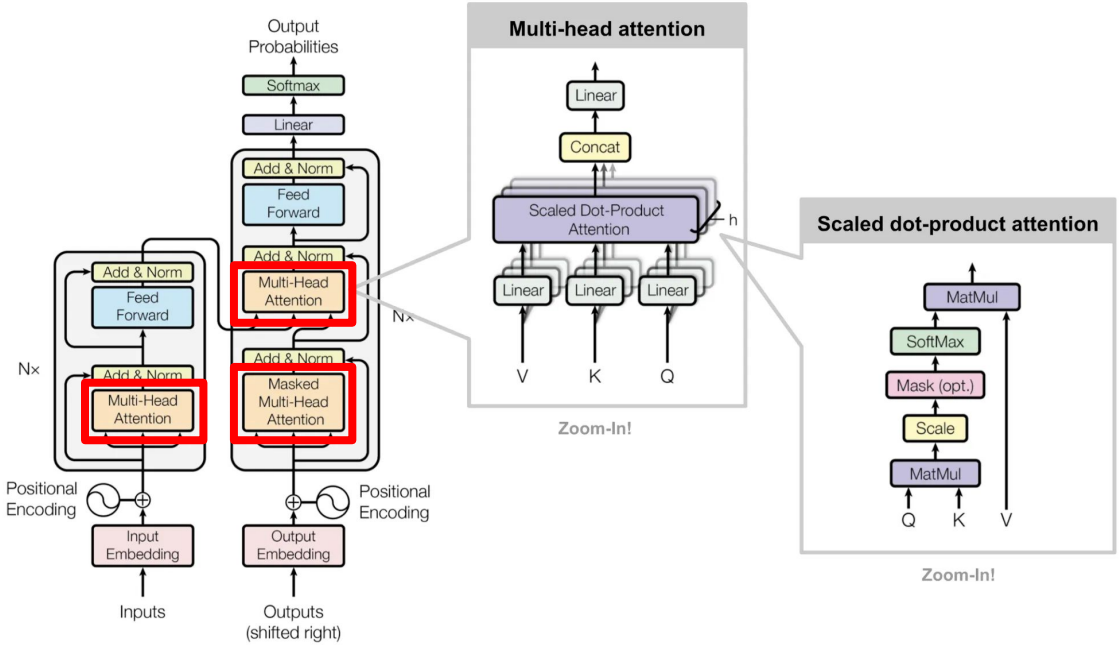
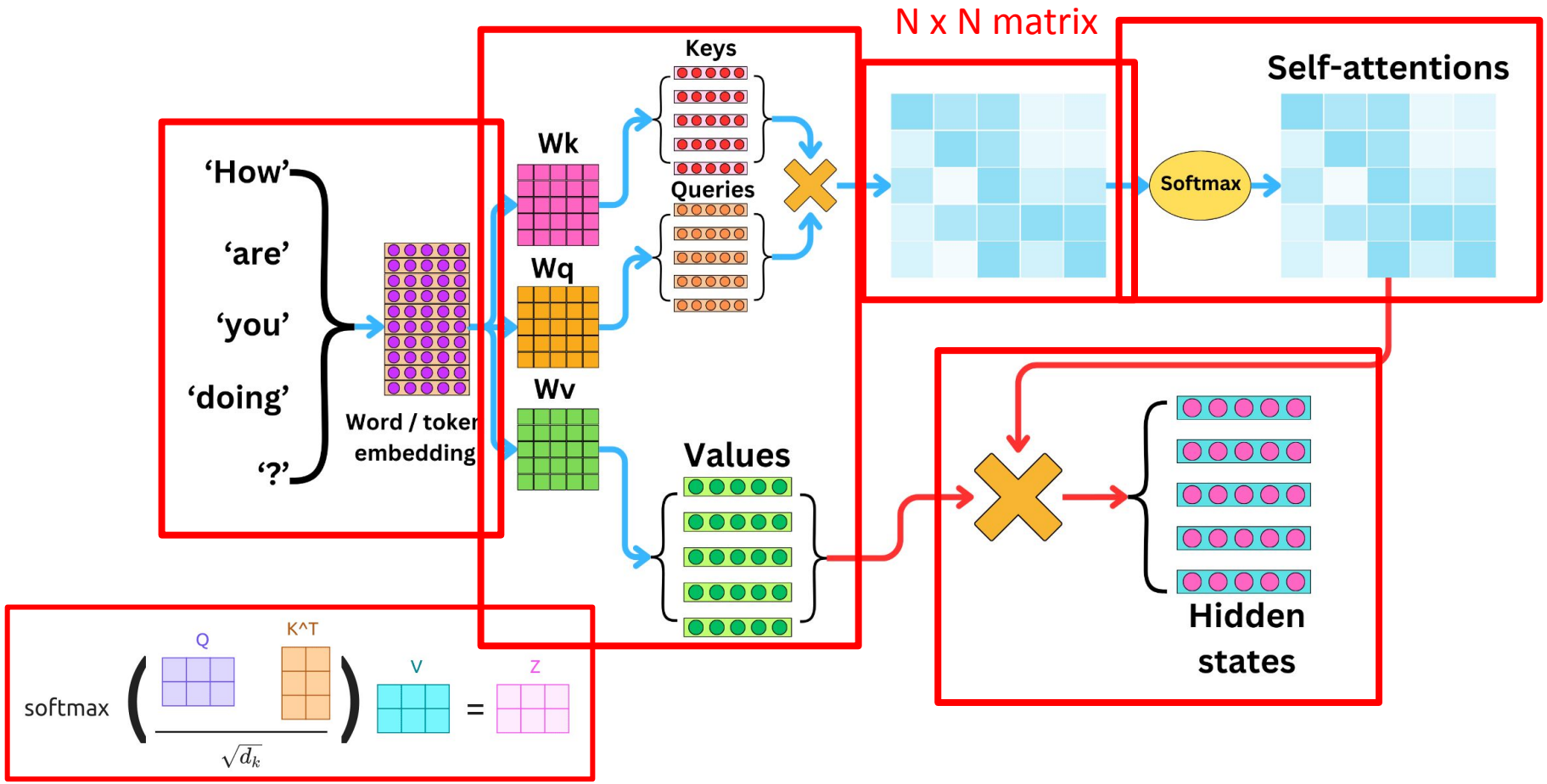
The Issue With Longer Contexts

What powers longer context?
Self-attention!

Scaling context is hard:
self-attention is quadratic in sequence length



⚠️ **Self-attention: time & memory complexity is $O(N^2)$ in sequence length N**



Source: https://newsletter.theaiedge.io/p/understanding-the-self-attention

Prior Work To Speed Attention

What they do

- Sparse attention (local windows, strided)
- Low-rank approximation
- Theoretically attractive
- Focus on reducing FLOPs:
  O(N) or O(N log N)

Why they fail in practice

- Ignore IO cost entirely
- No real wall-clock speedup
  - FLOPs are not correlated with speed
- No widespread adoption

💡 **FLOPs don't tell the full story!**

If attention is **quadratic**,
is the **bottleneck**
compute OR ***memory movement***?

Compute-Bound vs Memory-Bound

Arithmetic Intensity = FLOPs per byte of memory access

💾 Memory-Bound

Time determined by memory access.
Computation is fast, but waiting for data.

Examples:

- Elementwise: activation, dropout
- Reduction: ***softmax***, layernorm, sum

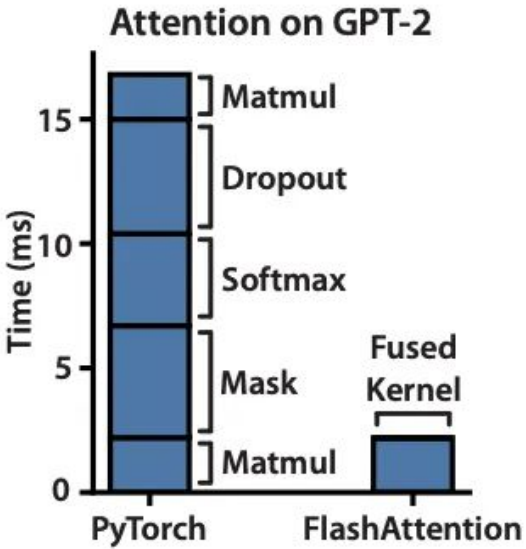
⚙️ Compute-Bound

Time determined by arithmetic ops.
HBM access is smaller bottleneck.

Examples:

- Matrix multiplication (large inner dim)
- Convolution (many channels)

The Real Bottleneck to Tackle



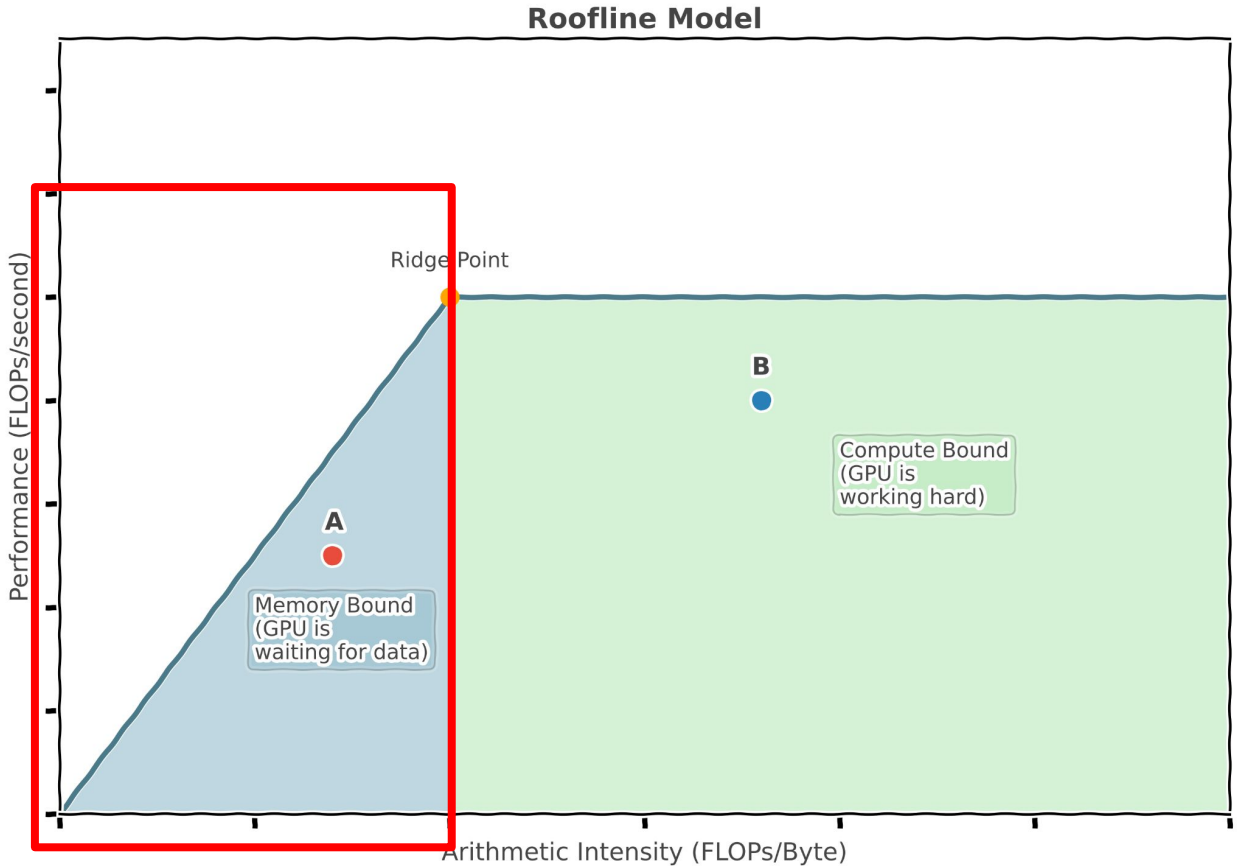
Modern GPUs make computations like matrix multiplication very fast.

BUT many operations in attention are **memory-bound** instead.
Reading/writing to GPU memory (HBM) dominates overall runtime.

This same bottleneck exists in databases, image processing, and many other fields.

💡 **FlashAttention tackles this memory movement bottleneck.**

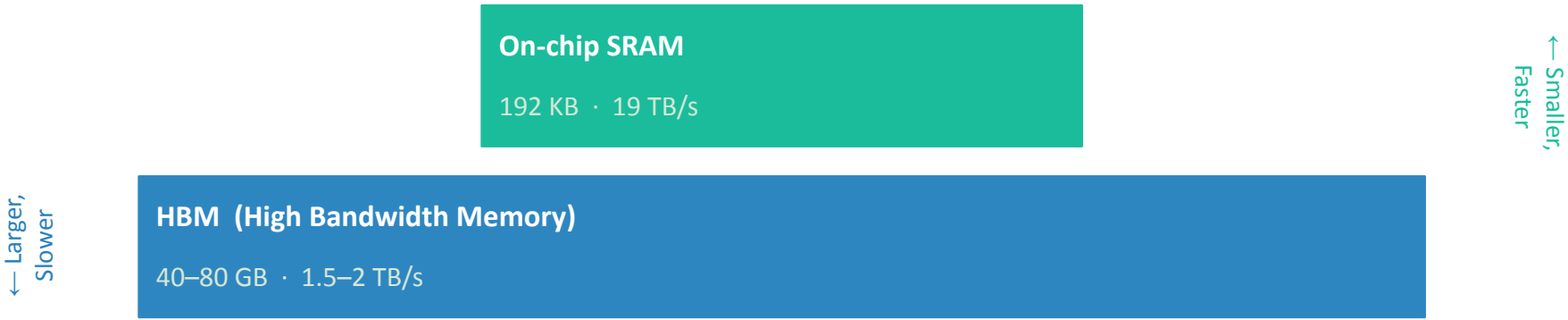
Compute-Bound vs Memory-Bound



Our focus!

GPU Memory Hierarchy

SRAM is 10× faster than HBM but MUCH tinier! (192 KB vs 40+ GB).
The goal: **keep data in SRAM as long as possible.**

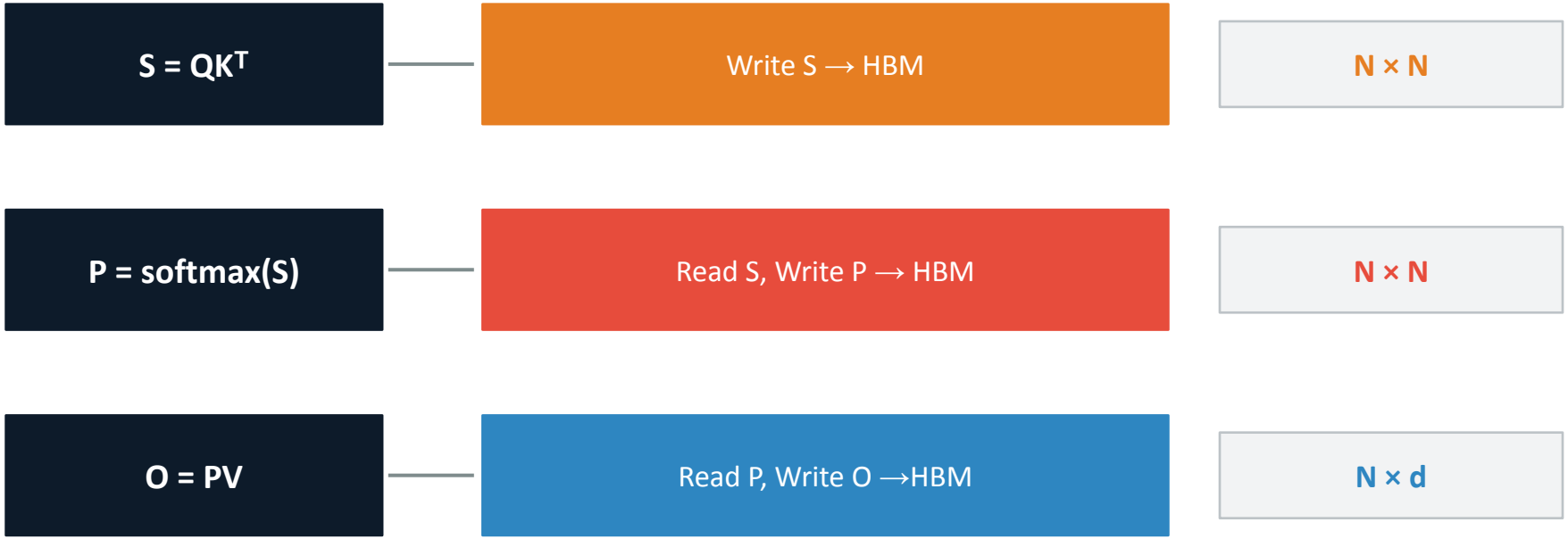


How GPU Kernels Execute

On GPUs, every operation runs as a kernel, so each kernel.:.

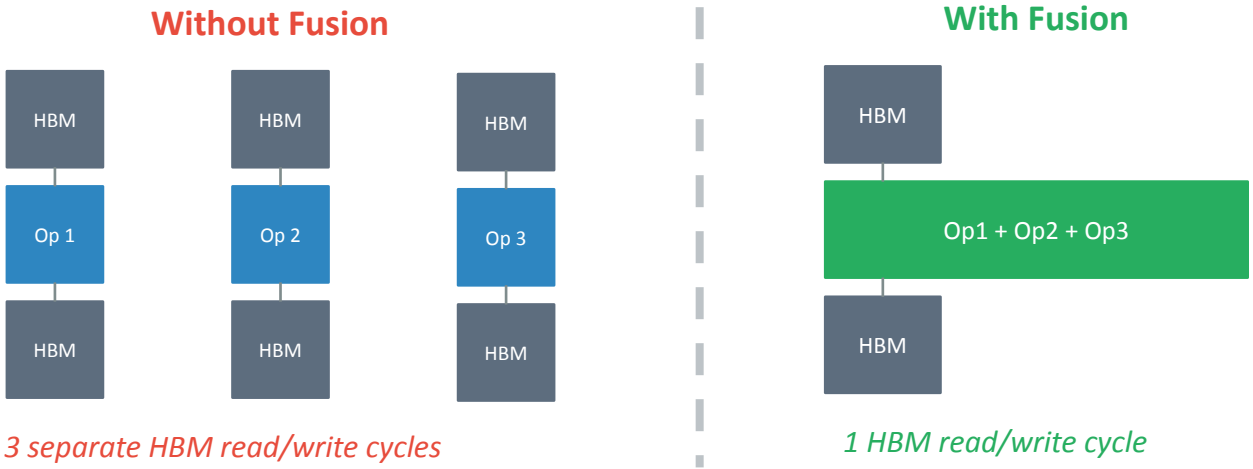
💡 **FlashAttention reduces how often we have to go back to HBM!**

How Memory Blows Up



S and P are both N×N → massive HBM traffic: write S, read S, write P, read P, write O

A Naive Fix: Kernel Fusion



⚠️ **But this doesn't work for attention!**

1. The attention matrix is too large to keep on-chip
2. Training requires storing intermediate matrices for the backward pass

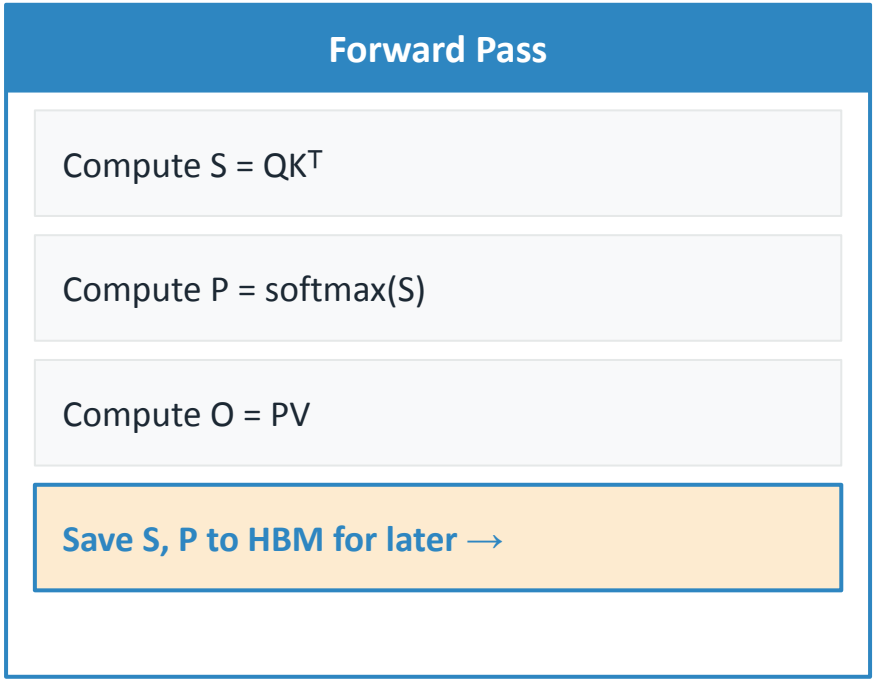
Why Can't We Just Discard S and P?

Inference:

1. Forward pass

Training:

1. **Forward pass**
   a. Store S, P to HBM
2. Compute loss
3. **Run backward pass to compute gradients**
   a. Load S, P from HBM
4. Update weights



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Backward Pass

Need P to compute $\partial L/\partial V$

Need S to compute $\partial L/\partial P$

Need P to compute $\partial L/\partial Q$, $\partial L/\partial K$

Reload S, P from HBM ←

Standard Attention: IO Complexity

HBM Accesses: $\Theta(Nd + N^2)$

Nd — Loading Q, K, V

Each of Q, K, V is $N \times d$
$\longrightarrow$ Linear in sequence length N

N^2 — Storing & Reading S, P

$S = QK^T$ and $P = \text{softmax}(S)$
both $N \times N \longrightarrow$ quadratic!

The N^2 term dominates for long sequences. This is what we must eliminate.

Can we compute
exact attention
without writing ***NxN*** matrices
to memory?

Yes, with FlashAttention!

Solution at a Glance

💡 **What is FlashAttention?**

An algorithm that computes **exact attention** (no approximation) with far fewer memory accesses.

✅ **Implementation**

Implemented in **CUDA** for fine-grained memory control.

Fuses all attention ops into one GPU kernel.

Results

2–3×

Faster attention computation
=> faster training of existing methods

O(N)

Lower bound for exact attention
=> enables training of larger models

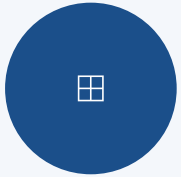
Extendible

Useful primitive for other approximate attention algorithms

Methodology

FlashAttention: Two Ideas, One Elegant Solution

Goal: compute exact attention without materializing the N×N matrix in slow memory



Tiling

Split Q, K, V into small blocks that fit in fast SRAM. Compute attention tile-by-tile entirely on-chip — never forming the full N×N matrix.

Works trivially for matmul

Online Softmax

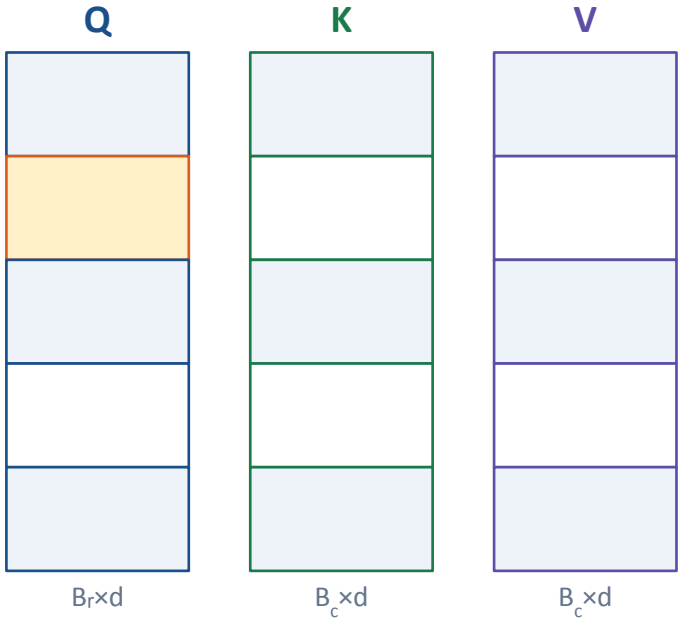
Track running max m and sum ℓ as we process each tile. Merge statistics incrementally with rescaling — exact softmax without seeing the full row.

Solves the softmax tiling problem

Together → One fused CUDA kernel · O(N) memory · Exact result · 2–4× faster

Tiling: Compute Attention Block by Block

Q, K, V split into blocks:



load block

On-chip SRAM — 19 TB/s · ~20 MB

tile ($B_r \times B_c$)

GPU HBM — 1.5 TB/s · 40 GB

Q, K, V, O (full matrices live here)

Block sizes: $B_c = \lceil M/4d \rceil$ and $B_r = \min(\lceil M/4d \rceil, d)$ — chosen so 4 blocks fit in SRAM of size M. That's it.

Matmul tiles trivially — partial sums just accumulate. But softmax needs the full row. So what do we do?

Tiling: Why Softmax Breaks It

$$\text{softmax}(x)_i = \exp(x_i) \,/\, \Sigma_j \exp(x_j) \quad \leftarrow \textit{needs ALL j simultaneously}$$

What happens if we naïvely softmax each tile separately (row of N attention scores, split into T_c tiles of B_c each)?

Tile 1

x_1 x_2 x_3

local denom:
$\exp(x_1 - \tilde{m}) + \exp(x_2 - \tilde{m}) + \exp(x_3)$

Tile 2

x_4 x_5 x_6

local denom:
$\exp(x_4 - \tilde{m}) + \exp(x_5 - \tilde{m}) + \exp(x_6 - \tilde{m})$

Tile 3

x_7 x_8 x_9

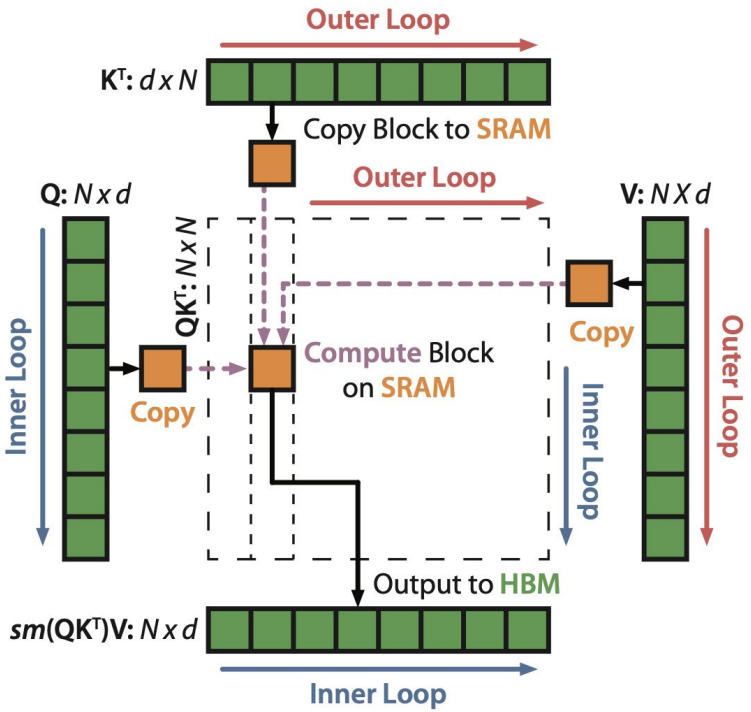
local denom:
$\exp(x_7 - \tilde{m}) + \exp(x_8 - \tilde{m}) + \exp(x_9 - \tilde{m})$

✗ **Each tile normalizes by its own local sum** → the three softmax outputs use different denominators and cannot be combined.

Softmax couples ALL elements of a row. You can't tile it naively — there's no fix after the fact.

Tiling: Loop Structure

Outer loop iterates over K/V blocks (j) — inner loop iterates over Q blocks (i)



For each K/V block j (outer), every Q block i (inner) computes tile S_{ij} on SRAM — merged online, written to HBM.

What Fits in SRAM?

B_r

Q Block Rows

Number of query rows loaded at once into SRAM

B_c

K, V Block Cols

Number of key/value rows loaded alongside each Q tile

M

SRAM Budget

Total on-chip SRAM — the constraint everything must fit within

Block size constraint: $2B_r \times d + 2B_c \times d \leq M \quad \rightarrow \quad B_c = \lceil M / 4d \rceil, B_r = \min(\lceil M / 4d \rceil, d)$

What lives on-chip simultaneously:

Try it yourself!

https://andrewma5.github.io/flashattention-softmax-demo/

Why Online Softmax is Non-Trivial

✗ Naïve Attempt

```
For each tile j:

  softmax_j = exp(S_ij - ~m_ij) / rowsum(exp(S_ij - ~m_ij))
```

$O_{\square} = \Sigma_{\square}\ \text{softmax\_j} \cdot V_{\square} \leftarrow$ incompatible scale

Why it fails:
softmax_1 is normalized by Σ(tile 1)
softmax_2 is normalized by Σ(tile 2)

These are different denominators. The outputs live on incompatible scales — you cannot add them.

✓ What We Need

A way to process tiles one at a time and end up with the correct globally-normalized result:

1. See tile 1, get partial result $\tilde{O}_1$
2. See tile 2, update $\tilde{O}_1 \rightarrow \tilde{O}_2$ using tile 2 info
3. … Continue for all T_c tiles
4. ✓ Final result = softmax(QK^T)V ✓

Key: carry (m, ℓ) as running stats, rescale at each step

The challenge: find statistics you can carry across tiles that enable exact global normalization after the fact.

Online Softmax: The Solution

Numerically Stable Softmax

m(x)	= max x_i	*row max*
$f(x)_i$	= $\exp(x_i - m(x))$	*shifted exp*
$\ell(x)$	= $\Sigma_i f(x)_i$	*normalizer*
softmax(x)	= f(x) / ℓ(x)	*output*

Block Decomposition ($x^{(1)} \cup x^{(2)} \rightarrow x$)

m(x)	= max($m(x^{(1)})$, $m(x^{(2)})$)	*merge maxes*
$\ell(x)$	= $\ell(x) = e^{(m^{(1)}-m)} \cdot \ell^{(1)} + e^{(m^{(2)}-m)} \cdot \ell^{(2)}$	*rescale*
softmax(x)	= f(x) / ℓ(x)	*exact* ✓

Key: carry only **(m, ℓ)** per row → merge any two blocks exactly → one block at a time

Track only (m, ℓ) per row → exact softmax one block at a time, never needing the full row.

What Is Stored Per Row

m

Running Max

Maximum attention score seen so far in this row. Keeps exp() from overflowing. Updated each block.

ℓ

Running Sum

Sum of all exp(s − m) values seen so far. The normalization denominator. Rescaled when m changes.

O

Partial Output

Accumulated weighted sum of V rows so far. Rescaled each block. Final answer after all tiles.

Update rule per new block j:

$$m_{new} = \max(m_i, m_{ij}) \qquad \ell_{new} = e^{m_i - m_{i\,new}} \cdot \ell_i + e^{m_{ij} - m_{i\,new}} \cdot \ell_{ij} \qquad O_{new} = \left(\ell_i \cdot e^{m_i - m_{i new}} \cdot O_i + e^{m_{ij} - m_{i new}} \cdot \tilde{P}_{ij} \cdot V_j \right) / \ell_{new}$$

Numeric Example: Setup

N=4, d=1, B^c=2 → 2 tiles of size 2

Q, K split into blocks first. Each tile S_i□ = Q_i × K□^{T} computed on SRAM:

Tile j=1	Tile j=2
$Q_1 \times K_1^T$ on SRAM	$Q_1 \times K_2^T$ on SRAM
[1.0, 3.0]	[2.0, 0.5]

← *Outer loop j=1 first, then j=2* →

Running stats before we start:

m_i (running max)	ℓ_i (running sum)	O_i (output so far)
−∞	0	0

Key question: can we get exact softmax([1.0, 3.0, 2.0, 0.5])·V without ever seeing the full row at once?

Tile j=1: Local Statistics

Load [1.0, 3.0] into SRAM — compute local softmax stats

Tile j=1 = [1.0, 3.0]

1 **Local max — for numerical stability**

$\tilde{m}_{i1}$ = max(1.0, 3.0) = **3.0**

→ *Subtract this from all scores before exp → prevents overflow*

2 **Shift and exponentiate**

e^(1.0 − 3.0) = e^(−2.0) = **0.135**

e^(3.0 − 3.0) = e^(0.0) = **1.0**

→ *These are the unnormalized attention weights $\tilde{P}_{i1}$ for this tile*

3 **Local sum — tile's own denominator**

$\tilde{\ell}_{i1}$ = 0.135 + 1.0 = **1.135**

→ *This is only the denominator for tile 1 — not the global denominator yet*

After tile 1: local stats are $\tilde{m}_{i1}$ = 3.0, $\tilde{P}_{i1}$ = [0.135, 1.0], $\tilde{\ell}_{i1}$ = 1.135 — all tile-local, not global yet.

Tile j=1: Update Running Statistics

Merge tile 1 stats into global running stats — first iteration is trivial

Before tile 1:

m_i	ℓ_i	O_i
$-\infty$	0	0

New global max

$m_{new} = \max(-\infty, 3.0) = 3.0$

→ *First tile so trivially takes the tile's max*

New normalizer

$\ell_{new} = e^{(-\infty-3.0)} \cdot 0 + e^{(3.0-3.0)} \cdot 1.135 = 0 + 1.0 \cdot 1.135 = 1.135$

→ *Old ℓ_i=0 contributes nothing; new tile's $\tilde{\ell}_{i1}$ = 1.135 carries over directly*

Partial output after tile 1

$O^{(1)} = (0.135 \cdot V_1 + 1.0 \cdot V_2) / 1.135$

→ *Normalized by ℓ_i^{new}=1.135 — correct for tile 1 only, will be rescaled when tile 2 arrives*

Running stats after tile 1:

$m_i = 3.0$	$\ell_i = 1.135$	$O_i = O_i^{(1)}$

Now we load tile 2. The running stats (m, ℓ, O) are all we carry forward — no tile 1 data kept in SRAM.

Tile j=2: Local Statistics

Load [2.0, 0.5] into SRAM — compute local softmax stats

Tile j=2 = [2.0, 0.5]

1 **Local max**

$\tilde{m}_{i2} = \max(2.0, 0.5) = 2.0$

→ *Different from $m_i = 3.0$ — this is exactly the incompatibility from the previous slides*

2 **Shift by local max and exponentiate**

$e^{(2.0 - 2.0)} = e^{(0.0)} = 1.0$

$e^{(0.5 - 2.0)} = e^{(-1.5)} = 0.223$

→ *$\tilde{P}_{i2} = [1.0, 0.223]$ — unnormalized weights for tile 2*

3 **Local sum**

$\tilde{\ell}_{i2} = 1.0 + 0.223 = 1.223$

→ *Tile 2's local denominator — normalized by $\tilde{m}_{i2}=2.0$, not the global max m=3.0*

$\tilde{m}_{i2} = 2.0 \neq m_i = 3.0$ — the running max from tile 1 is larger. Online softmax handles this with rescaling.

Tile j=2: Merge Stats — Online Softmax

This is the key step — rescale to the new global max

Running stats coming in from tile 1:

$m_i = 3.0$ | $\ell_i = 1.135$

New global max

$m_{new} = \max(3.0, 2.0) = 3.0$ ← unchanged

→ *Tile 2's max (2.0) is smaller than the running max (3.0) — no change needed*

Tile j=2: Merge Stats — Online Softmax

This is the key step — rescale to the new global max

Running stats coming in from tile 1:

$m_i = 3.0$ | $\ell_i = 1.135$

New global max

$m^{new} = \max(3.0, 2.0) = 3.0$ ← unchanged

→ *Tile 2's max (2.0) is smaller than the running max (3.0) — no change needed*

Rescale old ℓ_i to new max

$e^{(3.0-3.0)} \cdot 1.135 = 1.0 \cdot 1.135 = 1.135$

→ *Since max didn't change, $e^{(m_i - m_i^{new})} = e^0 = 1.0$ — old ℓ_i carries over unchanged*

Scale new tile's $\tilde{\ell}_{i2}$ to global max

$e^{(2.0-3.0)} \cdot 1.223 = 0.368 \cdot 1.223 = 0.450$

→ *$e^{(\tilde{m}_{i2} - m_i^{new})} = e^{(2.0-3.0)} = 0.368$ — corrects tile 2's sum to be consistent with global max 3.0*

Tile j=2: Merge Stats — Online Softmax

This is the key step — rescale to the new global max

Running stats coming in from tile 1:

$m_i = 3.0$ | $\ell_i = 1.135$

New global max

$m_{new} = \max(3.0, 2.0) = 3.0$ ← unchanged

→ *Tile 2's max (2.0) is smaller than the running max (3.0) — no change needed*

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Scale new tile's $\tilde{\ell}_{i2}$ to global max

$e^{(2.0-3.0)} \cdot 1.223 = 0.368 \cdot 1.223 = 0.450$

→ *$e^{(\tilde{m}_{i2} - m_i^{new})} = e^{(2.0-3.0)} = 0.368$ — corrects tile 2's sum to be consistent with global max 3.0*

New global normalizer

$\ell_{new} = 1.135 + 0.450 = 1.585$

→ *This is the exact softmax denominator for the full row [1.0, 3.0, 2.0, 0.5]*

$\ell_i^{new} = 1.585$ is the exact global denominator — computed without ever seeing all 4 scores simultaneously.

Tile j=2: Update Output

Rescale old output, add new tile contribution, normalize

① **Rescale old $O_i^{(1)}$ to new global max**

$$e^{(m^{(1)} - m^{new})} \cdot O^{(1)} = e^{(3.0-3.0)} \cdot O^{(1)} = 1.0 \cdot O^{(1)}$$

→ *Max didn't change so e^0 = 1.0 — $O_i^{(1)}$ carries over with no modification*

Tile j=2: Update Output

Rescale old output, add new tile contribution, normalize

① **Rescale old $O_i^{(1)}$ to new global max**

$$e^{(m_{i1} - m_i^{new})} \cdot O_i^{(1)} = e^{(3.0 - 3.0)} \cdot O_i^{(1)} = 1.0 \cdot O_i^{(1)}$$

→ *Max didn't change so e^0 = 1.0 — $O_i^{(1)}$ carries over with no modification*

② **New tile contribution — scaled to global max**

$$e^{(\tilde{m}_{i2} - m_i^{new})} \cdot \tilde{P}_{i2} \cdot V = e^{(2.0 - 3.0)} \cdot [1.0,\ 0.223] \cdot V = 0.368 \cdot \tilde{P}_{i2} \cdot V$$

→ *$e^{(\tilde{m}_{i2} - m_i^{new})} = 0.368$ rescales tile 2's weights to be consistent with global max*

Tile j=2: Update Output

Rescale old output, add new tile contribution, normalize

① **Rescale old $O_i^{(1)}$ to new global max**

$$e^{(m^{(1)} - m^{new})} \cdot O^{(1)} = e^{(3.0-3.0)} \cdot O^{(1)} = 1.0 \cdot O^{(1)}$$

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② **New tile contribution — scaled to global max**

$$e^{(\tilde{m}_{i2} - m^{new})} \cdot \tilde{P}_{i2} \cdot V = e^{(2.0-3.0)} \cdot [1.0,\ 0.223] \cdot V = 0.368 \cdot \tilde{P}_{i2} \cdot V$$

→ *$e^{(\tilde{m}_{i2} - m_i^{new})} = 0.368$ rescales tile 2's weights to be consistent with global max*

③ **Combine and normalize by ℓ_i^{new}**

$$O^{new} = (\ell^{(1)} \cdot 1.0 \cdot O^{(1)} + 0.368 \cdot P_{i2} \cdot V) / 1.585$$

→ *$\ell_i = 1.135$ weights the old output, $0.368 \cdot \tilde{P}_{i2}$ weights the new tile*

$$O_i^{new} = \mathrm{diag}(\ell_i^{new})^{-1} \left(\mathrm{diag}(\ell_i) \cdot e^{(m_i - m_i^{new})} \cdot O_i + e^{(\tilde{m}_{ij} - m_i^{new})} \cdot \tilde{P}_{ij} V_j \right)$$

After both tiles: O_i^{new} is the exact attention output for this query row — S was never written to HBM.

Verify: Is the Answer Exact?

Compare online softmax result vs standard softmax on full row

Standard Softmax (full row)

Subtract global max = 3.0:

exp(1.0−3.0) = 0.135

exp(3.0−3.0) = 1.0

exp(2.0−3.0) = 0.368

exp(0.5−3.0) = 0.082

Sum = 0.135+1.0+0.368+0.082

= 1.585

Online Softmax (our result)

After tile 1:

$\ell_1 = 1.135, \quad m_1 = 3.0$

After tile 2:

rescale: $0.368 \cdot 1.223 = 0.450$

$\ell_{new} = 1.135 + 0.450$

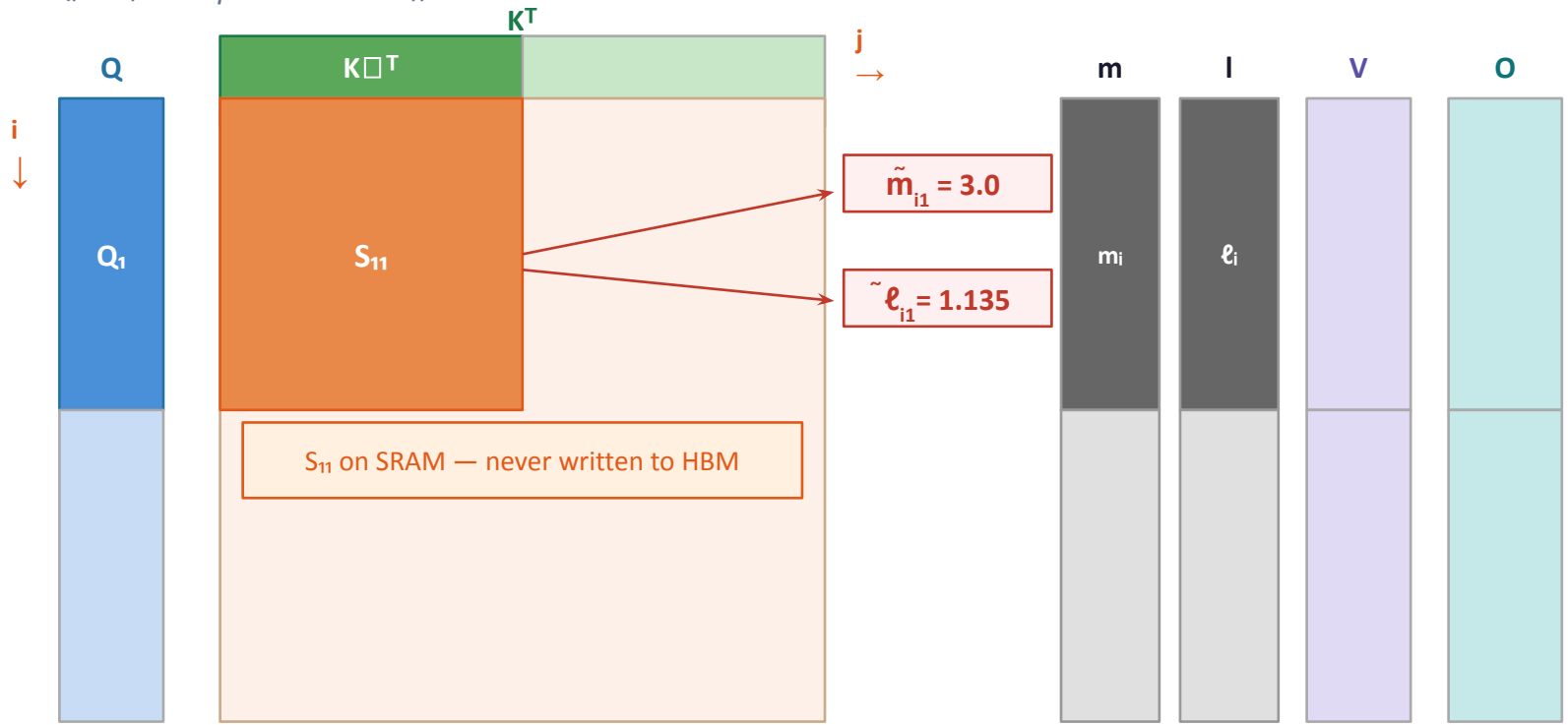
= 1.585 ✓

Both give $\ell = 1.585$ → same softmax weights → same output ✓

Exact match — not an approximation. Block size affects speed, not correctness. This is what Theorem 1 proves.

Tile j=1: What's Happening Spatially

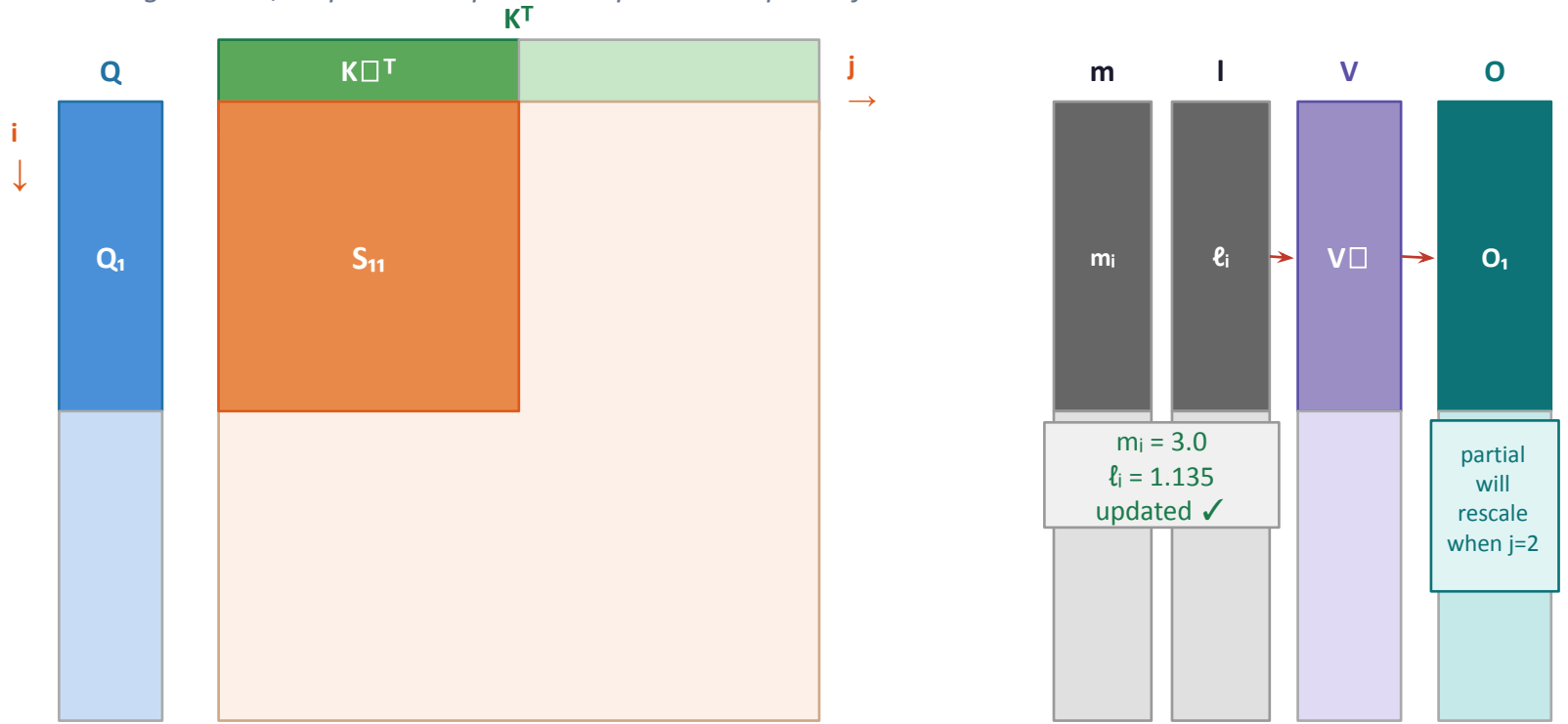
$Q_1 \times K_1^T$ computed as tile S_{11} on SRAM — local max and local sum extracted



$Q_1 \times K_1^T$ produces tile S_{11} on SRAM — local max $\tilde{m}_{i1}$ and local sum $\tilde{\ell}_{i1}$ extracted, S discarded.

Tile j=1: Partial Output Written

Running stats m_i, ℓ_i updated — partial output O_1 computed from S_{11} and written back to HBM

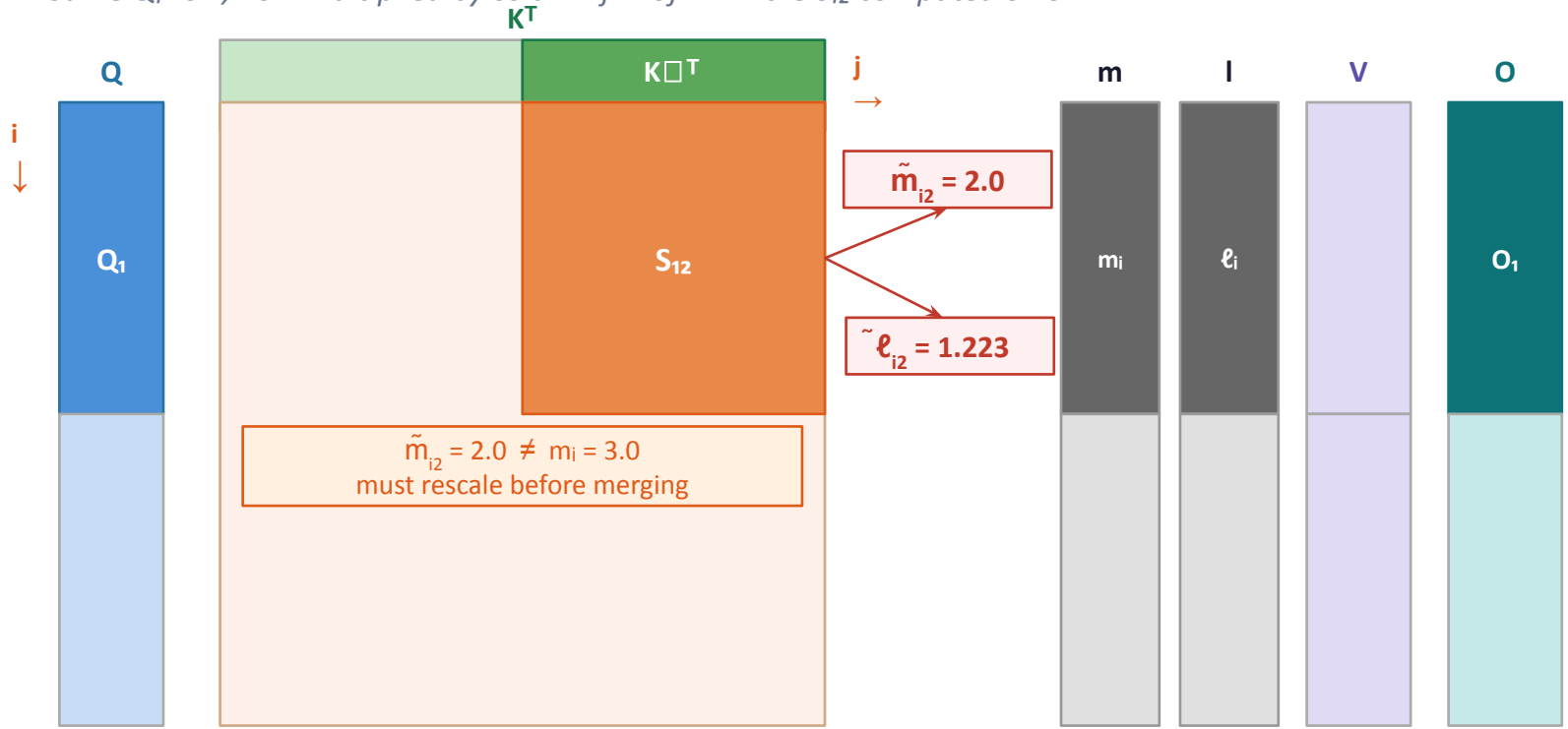


$$O_i^{(1)} = (0.135 \cdot v_1 + 1.0 \cdot v_2) \div 1.135$$

After tile j=1: S_{11} discarded from SRAM, m_i=3.0, ℓ_i=1.135, O_1 written to HBM.

Tile j=2: What's Happening Spatially

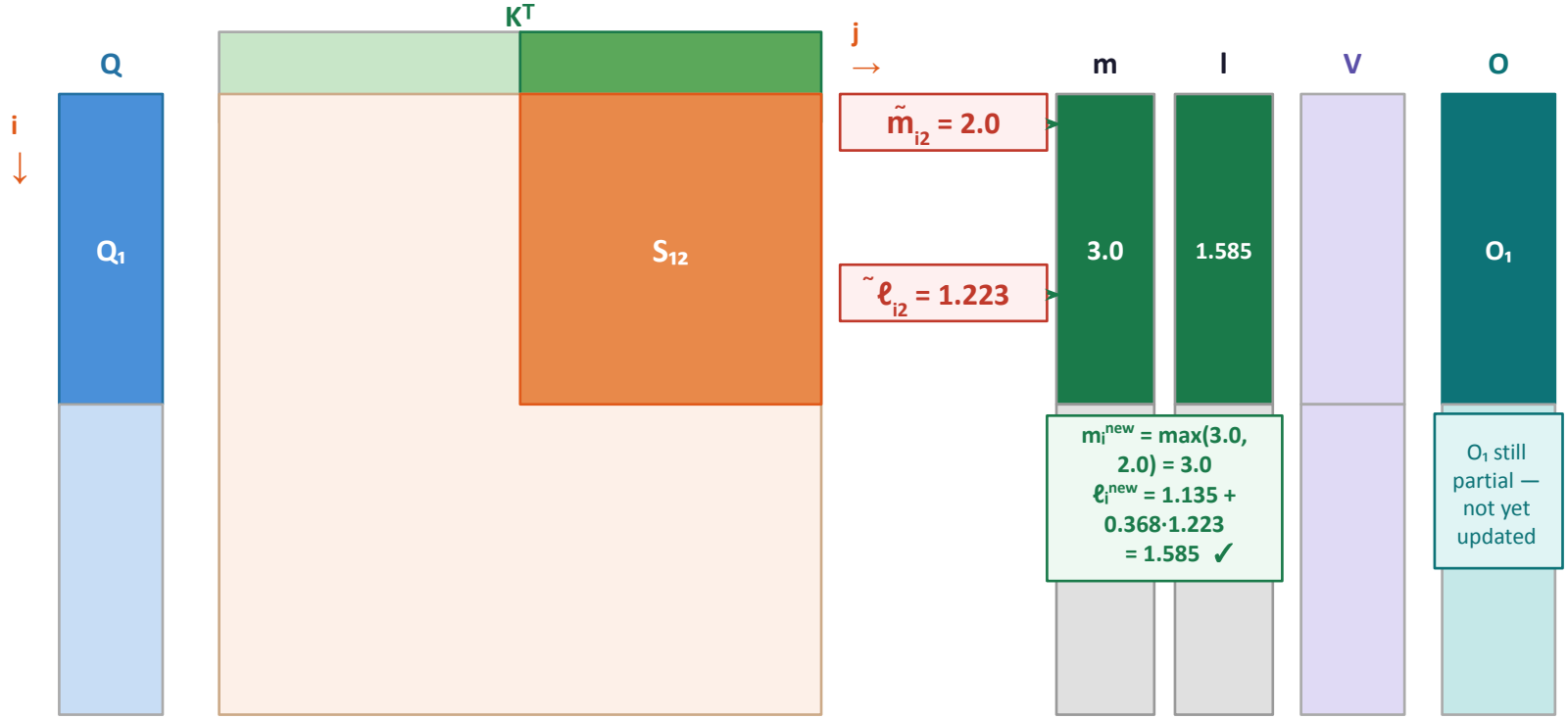
Same Q_1 row, now multiplied by column j=2 of K^T — tile S_{12} computed on SRAM



$Q_1 \times K_2^T$ produces S_{12} — local max $\tilde{m}_{i2}$=2.0 differs from running max m_i=3.0. Rescaling required.

Tile j=2: Merge Stats into Running Totals

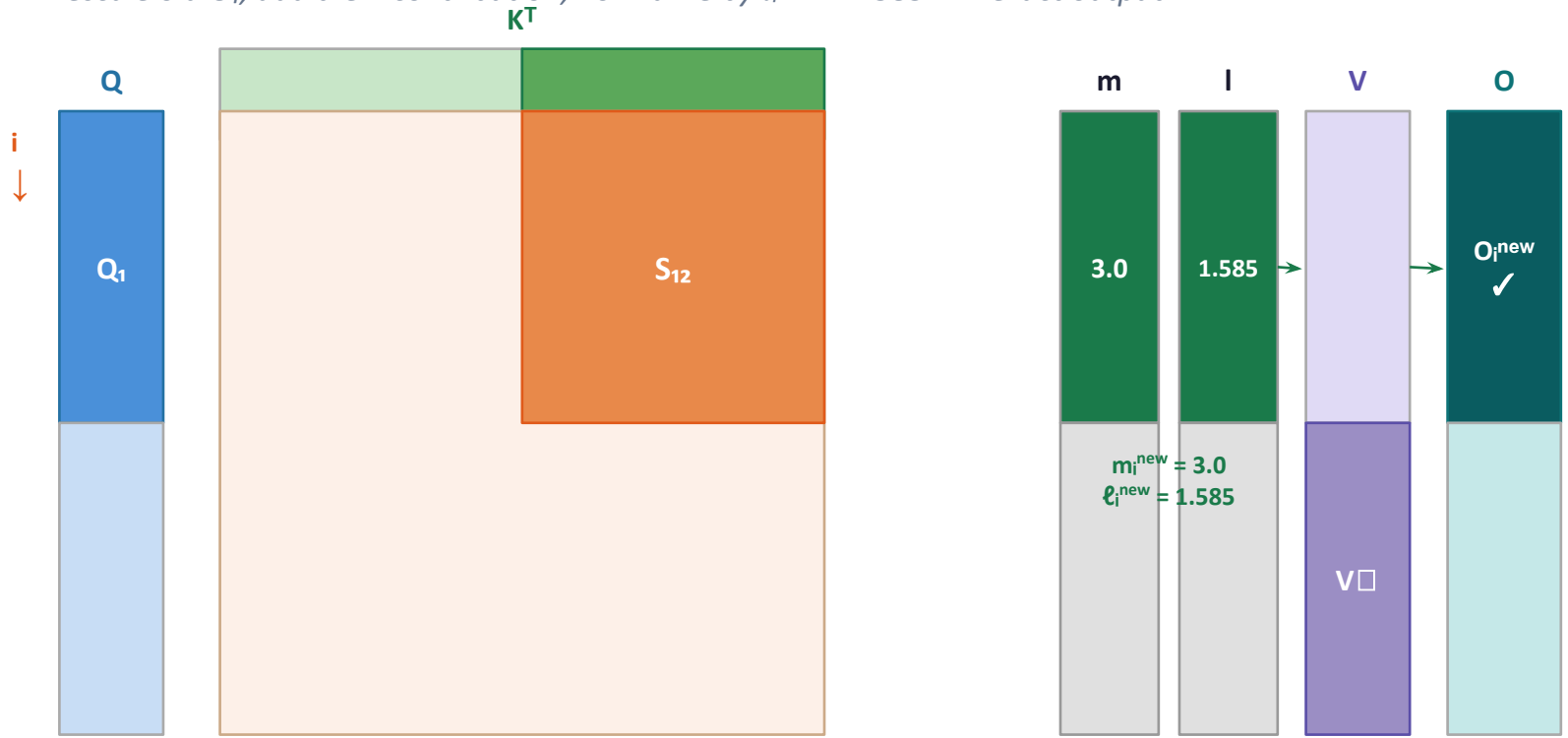
Rescale $\tilde{m}_{i2}$ and $\tilde{\ell}_{i2}$ to global max — write m_i^{new}=3.0 and ℓ_i^{new}=1.585 into m/l columns



$\tilde{m}_{i2}$=2.0 and $\tilde{\ell}_{i2}$=1.223 rescaled to global max 3.0 → merged into m_i^{new}=3.0, ℓ_i^{new}=1.585. O not yet updated.

Tile j=2: Final Output — Exact Result

Rescale old O_1, add tile 2 contribution, normalize by $\ell_i^{new} = 1.585 \longrightarrow$ exact output



$$O_i^{new} = \text{diag}(\ell_i^{new})^{-1} \cdot \left(\text{diag}(\ell_i) \cdot e^{(m_i - m_i^{new})} \cdot O_i + e^{(\tilde{m}_{ij} - m_i^{new})} \cdot \tilde{P}_{ij} V\square \right)$$

O_i^{new} is the exact attention output — S was never stored in HBM. This is what saves the memory.

What About Q_2?

Our numeric example only traces i=1. Here is the full 2×2 tile picture.

	j=1 (K_1^T)	j=2 (K_2^T)
i=1 (Q_1)	**S_{11}** $Q_1 \times K_1^T$ ✓ *covered in previous slides*	**S_{12}** $Q_1 \times K_2^T$ ✓ *covered in previous slides*
i=2 (Q_2)	**S_{21}** $Q_2 \times K_1^T$ *same process, independent of Q_1* *not shown*	**S_{22}** $Q_2 \times K_2^T$ *same process, independent of Q_1* *not shown*

Our numeric example traces the full i=1 row (S_{11} then S_{12}). Q_2 would repeat identically but independently.

Q_2 Repeats the Same Process Independently

No communication between query rows — each i has its own running stats

① **Own running stats**

Q_2 starts fresh with $m_i=-\infty$, $\ell_i=0$, $O_i=0$ — completely separate from Q_1

② **Same j sweep**

j=1 then j=2 — S_{21} and S_{22} computed, merged, output O_2 written to HBM row 2

③ **No shared state**

Q_1 and Q_2 never interact — this is why FlashAttention can parallelize across query blocks

④ **Memory stays bounded**

Only one tile of S ever lives on SRAM at a time, regardless of how many query rows there are

Each query row is self-contained — the online softmax runs independently per row, sweeping all K/V tiles.

Algorithm 1: FlashAttention — Full Picture

```
# Q,K,V ∈ ℝ□ˣ□  stored in HBM  →  output O ∈ ℝ□ˣ□

Init O□=0, ℓ□=0, m□=−∞  for all i

for j=1…T_c:  ▷ K,V tile loop
  Load K□,V□  HBM→SRAM
  for i=1…T_r:  ▷ Q tile loop
    Load Q□,O□,ℓ□,m□  HBM→SRAM
    S□□ = Q□K□□÷√d  ▷ on-chip
    m̃□□ = rowmax(S□□)
    P̃□□ = exp(S□□−m̃□□)
    ℓ̃□□ = rowsum(P̃□□)
    m□ⁿ□ᵂ = max(m□, m̃□□)
    ℓ□ⁿ□ᵂ = e^(m□−m□ⁿ□ᵂ)·ℓ□ + e^(m̃□□−m□ⁿ□ᵂ)·ℓ̃□□
    O□ⁿ□ᵂ = (ℓ□·e^(m□−m□ⁿ□ᵂ)·O□ + e^(m̃□□−m□ⁿ□ᵂ)·P̃□□V□) / ℓ□ⁿ□ᵂ
    Write O□,ℓ□,m□  SRAM→HBM
return O
```

Color Guide

- Init (O, ℓ, m)
- HBM ⟶ SRAM loads
- On-chip matmul (S_i□)
- Local softmax stats
- Online softmax + output
- Write back to HBM

① Init State

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,   ℓ□ = 0,   m□ = -∞  for all i
```

① Init State

1. **One row's state:** O_i, ℓ_i, m_i — initialized for every row of Q.
2. **No N×N matrix:** we only ever track O(N) values.
3. **$m_i = -\infty$:** ensures any real score will update the running max on the first tile.

② Outer Loop — Load K, V

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T_c:    ▷ K, V tile loop
   Load K□, V□    HBM → SRAM
```

② Outer Loop — Load K, V

1. **Outer loop over j:** iterates through T_c column blocks of K and V.
2. **Load once, reuse many:** K□ and V□ are loaded into SRAM once and held for all T_r inner iterations.
3. **Key savings:** this amortizes the HBM cost of K, V across every Q block.

③ Inner Loop — Load Q, O, ℓ, m

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T□:   ▷ K, V tile loop
   Load K□, V□   HBM → SRAM

   for i = 1…T_r:   ▷ Q tile loop
      Load Q□, O□, ℓ□, m□   HBM → SRAM
```

③ Inner Loop — Load Q, O, ℓ, m

1. **Inner loop over i:** iterates through T_r row blocks of Q.
2. **Load running state:** Q_i, O_i, ℓ_i, m_i come from HBM — these are the O(N) stats from prior iterations.
3. **Only 4 small loads per tile:** no full matrices ever loaded.

④ On-Chip Score Matrix

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T_c:    ▷ K, V tile loop
   Load K□, V□    HBM → SRAM

   for i = 1…T_r:    ▷ Q tile loop
      Load Q□, O□, ℓ□, m□    HBM → SRAM

      S□□ = Q□ K□□  ÷ √d    ▷ matmul on-chip
```

④ On-Chip Score Matrix

1. **$S_i□ \in \mathbb{R}^{BrxBc}$:** a small tile of the full N×N score matrix.
2. **Stays on SRAM:** computed entirely on-chip, never written to HBM.
3. **Heart of tiling:** standard attention writes all of N×N to HBM — FlashAttention never does.

💡 **S_{ij} and $\tilde{P}_{ij}$ are born and die in SRAM — they are never written to HBM.**

⑤ Local Softmax Statistics

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T□:   ▷ K, V tile loop
   Load K□, V□   HBM → SRAM

   for i = 1…T□:   ▷ Q tile loop
      Load Q□, O□, ℓ□, m□   HBM → SRAM

      S□□ = Q□ K□□ ÷ √d   ▷ matmul on-chip

      m̃□□ = rowmax(S□□)
      P̃□□ = exp(S□□ − m̃□□)
      ℓ̃□□ = rowsum(P̃□□)
```

⑤ Local Softmax Statistics

1. $\tilde{m}_{ij}$: local row max of this tile — used to shift exp for numerical stability.
2. $\tilde{P}_{ij}$: unnormalized attention weights for this tile — not yet globally correct.
3. $\tilde{\ell}_{ij}$: local sum — the denominator for this tile only.

💡 S_{ij} and $\tilde{P}_{ij}$ are born and die in SRAM — they are never written to HBM.

⑥ Rescale & Merge — Online Softmax

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T_c:    ▷ K, V tile loop
   Load K□, V□    HBM → SRAM

   for i = 1…T_r:    ▷ Q tile loop
      Load Q□, O□, ℓ□, m□    HBM → SRAM

      S□□ = Q□ K□□  ÷ √d    ▷ matmul on-chip

      m̃□□ = rowmax(S□□)
      P̃□□ = exp(S□□ − m̃□□)
      ℓ̃□□ = rowsum(P̃□□)

      m□ⁿᵉʷ = max(m□, m̃□□)
      ℓ□ⁿᵉʷ = e^(m□−m□ⁿᵉʷ) · ℓ□ + e^(m̃□□−m□ⁿᵉʷ) · ℓ̃□□
```

⑥ Rescale & Merge — Online Softmax

1. **m_i^{new}:** new global max = max of running max and this tile's max.
2. **Rescaling:** $e^{(m_i - m_i^{new})}$ corrects old ℓ_i; $e^{(\tilde{m}_{ij} - m_i^{new})}$ corrects new $\tilde{\ell}_{ij}$.
3. **Exact, not approximate:** this is the block-softmax decomposition — provably correct.

💡 S_{ij} and $\tilde{P}_{ij}$ are born and die in SRAM — they are never written to HBM.

⑦ Incremental Output Update

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T_c:    ▷ K, V tile loop
   Load K□, V□    HBM → SRAM

   for i = 1…T_r:    ▷ Q tile loop
      Load Q□, O□, ℓ□, m□    HBM → SRAM

      S□□ = Q□ K□□  ÷ √d   ▷ matmul on-chip

      m̃□□ = rowmax(S□□)
      P̃□□ = exp(S□□ − m̃□□)
      ℓ̃□□ = rowsum(P̃□□)

      m□ⁿᵉʷ = max(m□, m̃□□)
      ℓ□ⁿᵉʷ = e^(m□−m□ⁿᵉʷ) ·ℓ□ + e^(m̃□□−m□ⁿᵉʷ) ·ℓ̃□□

      O□ⁿᵉʷ = (ℓ□·e^(m□−m□ⁿᵉʷ) ·O□ + e^(m̃□□−m□ⁿᵉʷ) ·P̃□□V□) / ℓ□ⁿᵉʷ
```

⑦ Incremental Output Update

1. ① **Rescale old O_i:** multiply by $e^{(m_i - m_i^{new})}$ to adjust for updated global max.
2. ② **Add new tile:** $e^{(\tilde{m}_{ij} - m_i^{new})} \cdot \tilde{P}_{ij} V$□ — the highlighted yellow term was missing in the original slide!
3. ③ **Normalize:** divide by ℓ_i^{new} → exact result for all tiles seen so far.

💡 **S_{ij} and $\tilde{P}_{ij}$ are born and die in SRAM — they are never written to HBM.**

⑧ Write Back & Return

```
# Q,K,V ∈ ℝ□ˣ□  in HBM  →  output O ∈ ℝ□ˣ□

Init O□ = 0,  ℓ□ = 0,  m□ = −∞  for all i

for j = 1…T_c:    ▷ K, V tile loop
   Load K□, V□    HBM → SRAM

   for i = 1…T_r:    ▷ Q tile loop
      Load Q□, O□, ℓ□, m□    HBM → SRAM

      S□□ = Q□ K□□  ÷ √d   ▷ matmul on-chip

      m̃□□ = rowmax(S□□)
      P̃□□ = exp(S□□ − m̃□□)
      ℓ̃□□ = rowsum(P̃□□)

      m□ⁿᵉʷ = max(m□, m̃□□)
      ℓ□ⁿᵉʷ = e^(m□−m□ⁿᵉʷ) ·ℓ□ + e^(m̃□□−m□ⁿᵉʷ) ·ℓ̃□□

      O□ⁿᵉʷ = (ℓ□ ·e^(m□−m□ⁿᵉʷ) ·O□ + e^(m̃□□−m□ⁿᵉʷ) ·P̃□□V□)  /  ℓ□ⁿᵉʷ
```

⑧ Write Back & Return

1. **Write O_i, ℓ_i, m_i:** only O(N) scalars go back to HBM per inner iteration.
2. **S_{ij} and $\tilde{P}_{ij}$ discarded:** born and die on SRAM — never touch HBM.
3. **After all loops:** $O = \text{softmax}(QK^T)V$ exactly, with O(N) extra memory.

💡 **S_{ij} and $\tilde{P}_{ij}$ are born and die in SRAM — they are never written to HBM.**

Complexity Analysis : Define Variables

Variables

Symbol	Variable	Meaning
N	Sequence length	Number of tokens in the input sequence
d	Head dimension	Size of each query / key / value vector (e.g. 64 or 128)
M	Fast memory (SRAM)	Number of scalar floats that fit in on-chip GPU shared memory
Q, K, V	Input matrices	Query, Key, Value matrices ∈ ℝ^(N×d)
O	Output matrix	Attention output ∈ ℝ^(N×d)
B_n, B_r	Tiling block size	**B_n** — block size along columns (K/V dimension), **B_r** — block size along rows (Q dimension)

Why No N×N Matrix Is Ever Written

❌ Standard Attention

$S = QK^T$
→ *write N×N to HBM*

$P = \text{softmax}(S)$
→ *read + write N×N*

$O = PV$
→ *read N×N, write N×d*

HBM accesses: $O(Nd + N^2)$

VS

✅ FlashAttention

Load Q_i, K_j, V_j
SRAM tile load (HBM read)

Compute S_{ij} in SRAM
stays on-chip — never written to HBM

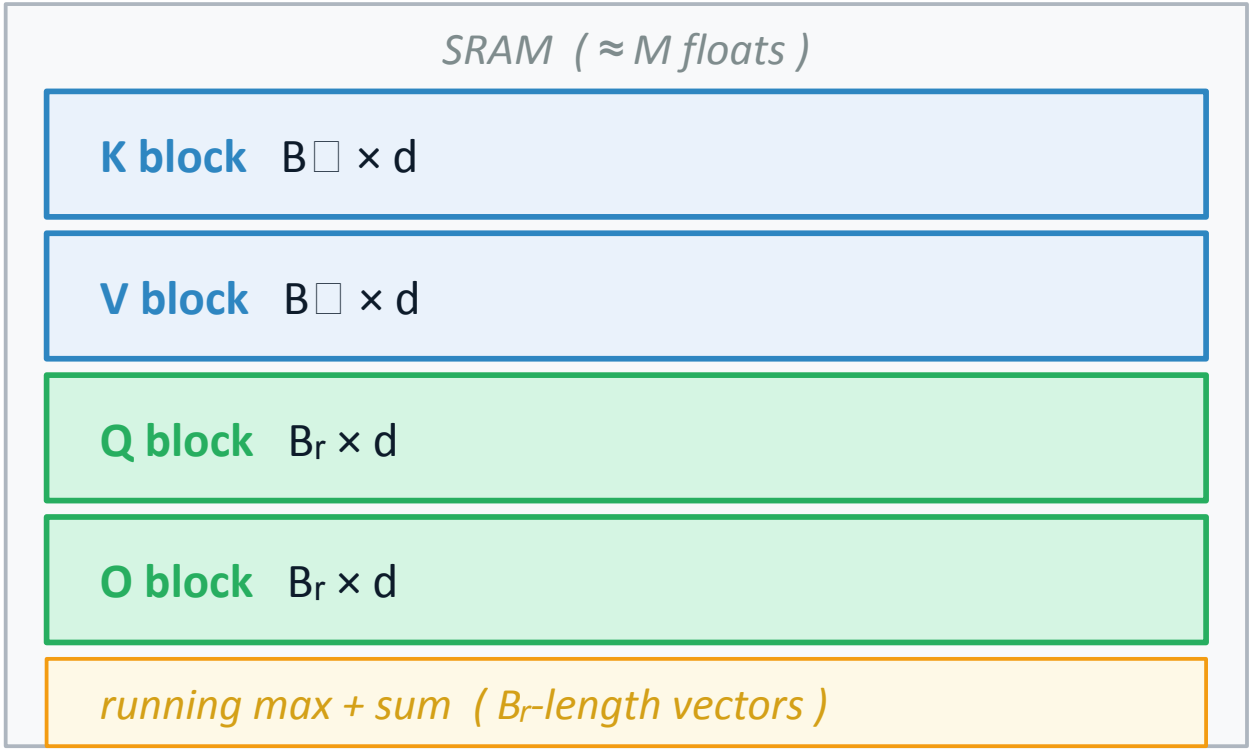
Update (m, ℓ, O) in SRAM
stays on-chip — never written to HBM

Write O_i once
written back each (i, j) iteration — S_{ij} never written

HBM accesses: $O(N^2d^2 / M)$ — far less!

FlashAttention: Why $\Theta(N^2d^2 / M)$

What lives in SRAM



SRAM constraint:

$$2B_{\square}d + 2B_r d + 2B_r \lesssim M$$

$$B_{\square}, B_r = \Theta(M/d)$$

FlashAttention: Why $\Theta(N^2d^2 / M)$

Where N^2d^2/M comes from

Outer loop over K/V blocks: N/B□ iterations

Inner loop over Q blocks: N/B_r iterations

Each step: load Q block ($B_r \times d$) from HBM

FlashAttention: Why $\Theta(N^2d^2 / M)$

Total Q reads = $(N/B_c)(N/B_r)(B_r d) = N^2 d / B_c$

Plug in $B_c = \Theta(M/d)$:

$N^2 d / (M/d) = \Theta(N^2 d^2 / M)$

Tight Lower Bound: FlashAttention Is Optimal

1

Must compute all N^2 dot-products $s_{ij} = q_i^T k_j$ for exact attention

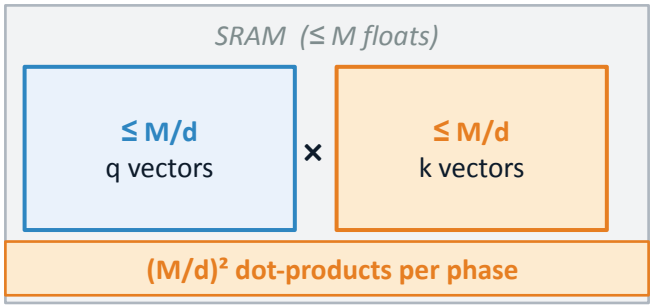
Tight Lower Bound: FlashAttention Is Optimal

2 Split execution into phases — each phase holds ≤ M scalars in SRAM

3 Each phase holds ≤ M/d query vectors and ≤ M/d key vectors

4 Dot-products per phase ≤ (M/d)(M/d) = $(M/d)^2$

Phase capacity



Tight Lower Bound: FlashAttention Is Optimal

5 Phases needed $\geq N^2 / (M/d)^2 = N^2d^2 / M^2$

6 Each phase moves $\Omega(M)$ data: needs fresh vectors from HBM

7 **Total I/O $\geq \Omega(N^2d^2/M^2) \cdot \Omega(M)$**

$$\Omega(N^2d^2 / M)$$

Lower bound matches FlashAttention: $\Theta(N^2d^2 / M)$ is optimal for exact attention **when $M \geq d^2$**

Gap in Proof?

6

Each phase moves $\Omega(M)$ data: needs fresh vectors from HBM

Block-Sparse FlashAttention

Core Idea

Restrict attention to a sparse mask — only compute scores s_{ij} where the mask allows. Skip entire K/V blocks that are fully masked.

Sparsity mask on N×N attention

Block-Sparse FlashAttention

IO Benefit

If fraction s of blocks are kept, I/O drops to $\Theta(s * N^2d^2 / M)$ — a factor of s^{-1} speedup over dense FlashAttention.

How it works

1. Determine a block sparsity mask $M_{ij} \in \{0,1\}$ for each (Q-block, K/V-block) pair
2. In the outer loop, skip any K/V block where mask = 0. No loads, no compute
3. Active blocks run identical FlashAttention kernel. Same tiling, same online softmax

When to use Block Sparse?

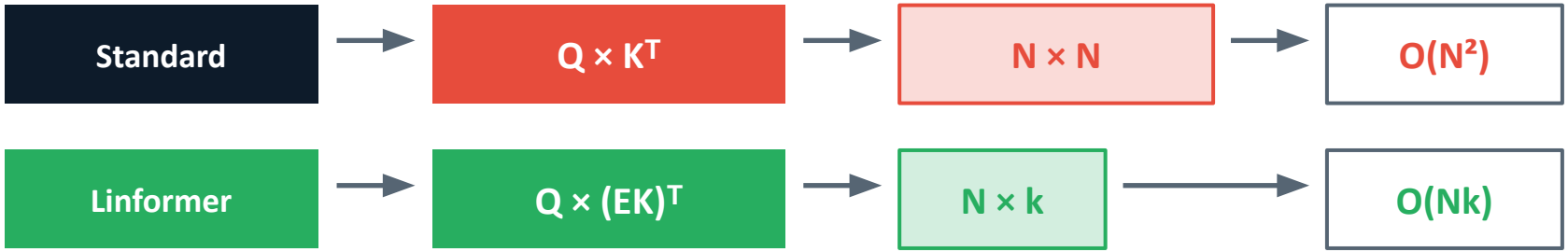
Approximation vs speed up

If you have a structured subset of blocks that approximates full attention

=> easily scale attention to longer sequences

Experiments

Linformer (2020)



E is a learned projection that compresses K, V from N → k rows (k ≪ N)

The catch: This is an approximation. Compressing K and V throws away fine-grained token interactions.

Reduces FLOPs on paper, but ignores memory IO cost → limited real-world speedup. No widespread adoption.

What FlashAttention Enabled

Faster Training

Before:

GPT-2 small: ~9.5 days (HuggingFace)

Even optimized Megatron-LM: ~4.7 days

With FlashAttention:

New BERT speed record (beat MLPerf)

GPT-2: up to 3.5× faster, same quality

Longer Context

Before:

Most models: ~2K token context

Approx. methods (Linformer) could reach ~64K, but quality suffered (not exact)

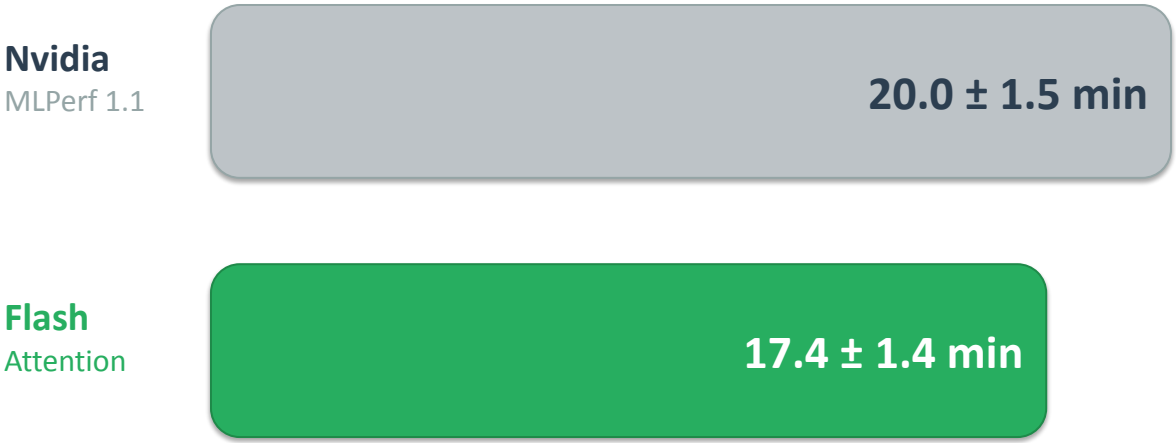
With FlashAttention:

Exact attention at 4K+ context

Better model quality, no approximation tradeoff

BERT Training Speed Record

BERT-large, 8×A100 GPUs, target 72.0% masked LM accuracy

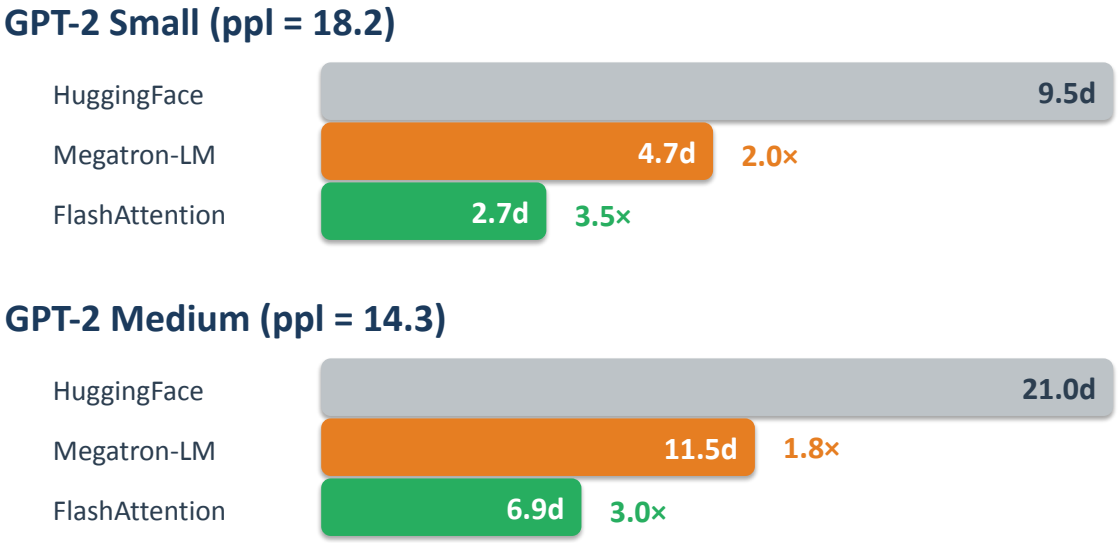


15% faster
than MLPerf record

Fastest single-node BERT training at time of publication
Same model, same hyperparameters, same target accuracy

GPT-2 Training Speedup

8×A100 GPUs, OpenWebText dataset



3× faster

than HuggingFace

Exact same perplexity

Same model quality, dramatically faster training

FlashAttention also enables longer context lengths to be trained effectively

Longer Context = Better Models

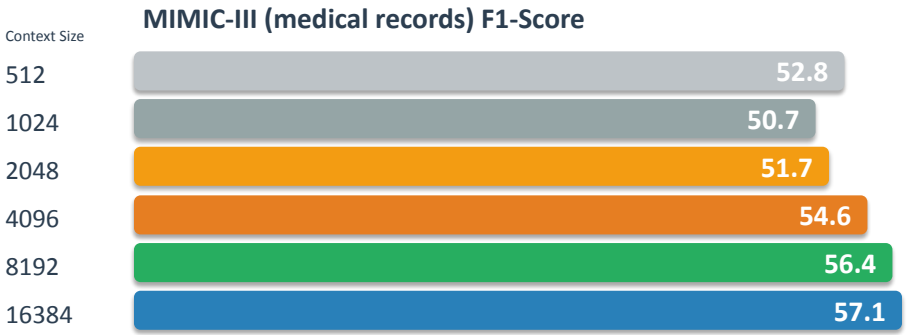
GPT-2 Small: Longer context, better perplexity

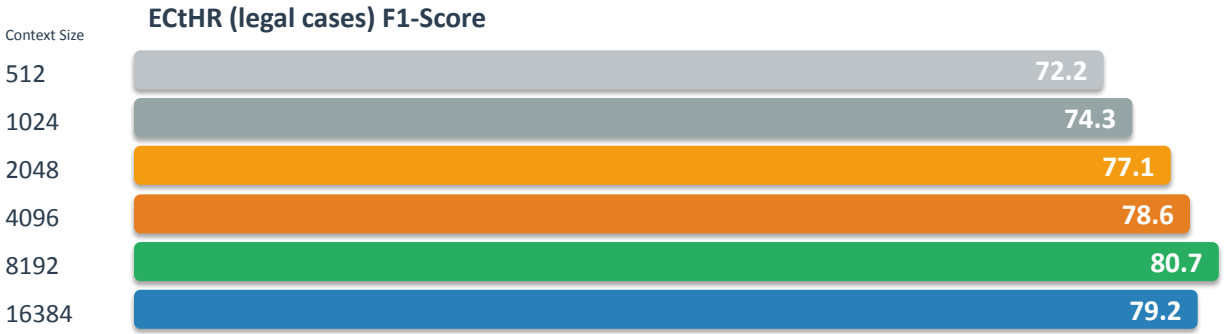
Implementation	Context	Perplexity	Speed
Megatron-LM	1K	18.2	4.7 days
FlashAttention	1K	18.2	2.7 days
FlashAttention	2K	17.6	3.0 days
FlashAttention	4K	**17.5**	3.6 days

4× longer context, 0.7 better perplexity, still 30% faster

Longer Context = Better Models

Long Document Classification Experiments

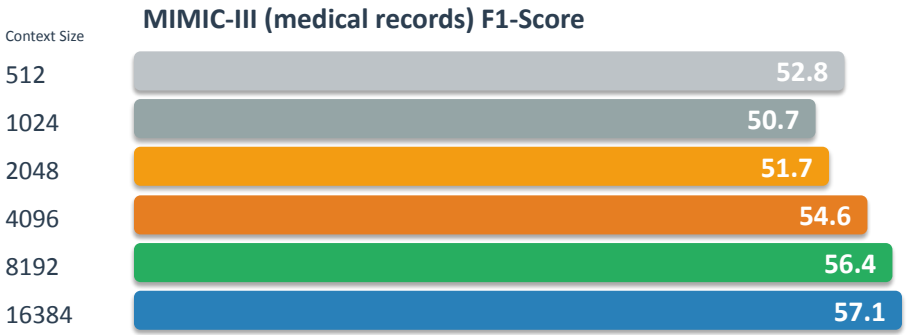


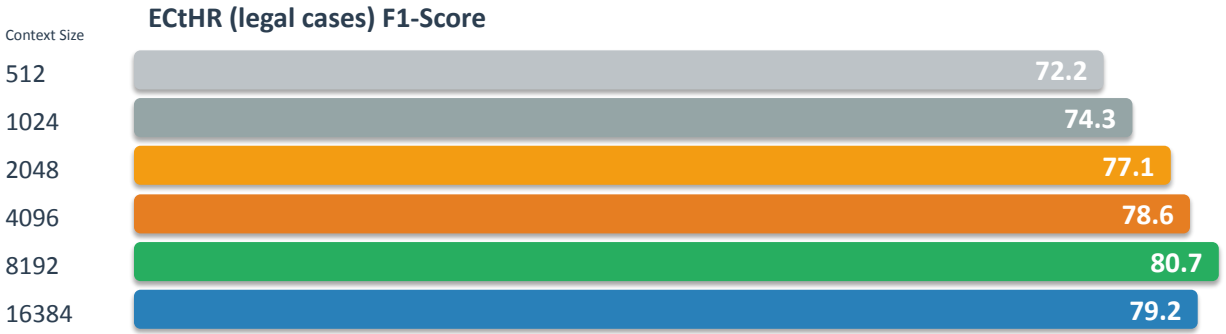


Longer Context models perform better on Long Document Classification

Longer Context = Better Models

Long Document Classification Experiments



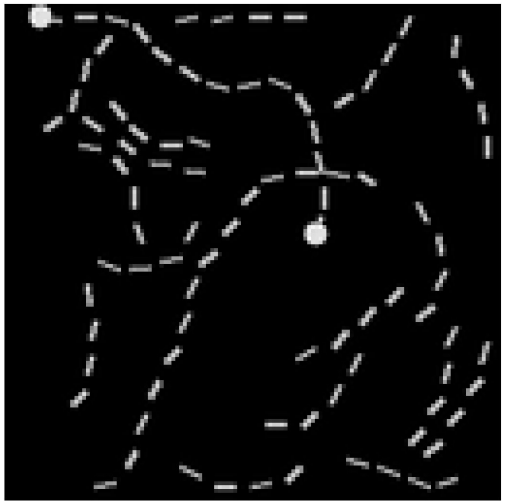


Experiment Design: Truncate document for smaller context windows.

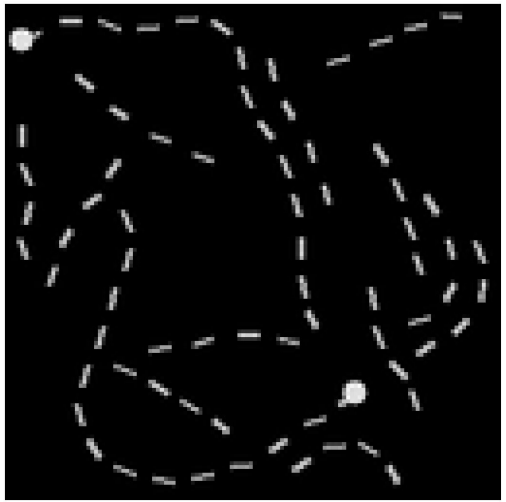
Is this valid?

Path-X and Path-256 Benchmarks

Test LLM ability to find long range spatial dependencies



Positive Example



Negative Example

Long Range Arena: A Benchmark for Efficient Transformers (Tay et. al, 2020)

First Transformer to Solve Path-X

Sequence length 16K — all prior Transformers failed

Model	Path-X (16K)	Path-256 (64K)
Transformer	✗ random	✗ random
Linformer	✗ random	✗ random
Linear Attention	✗ random	✗ random
Performer	✗ random	✗ random
Local Attention	✗ random	✗ random
Reformer	✗ random	✗ random
SMYRF	✗ random	✗ random
FlashAttention	✓ 61.4%	✗ random
Block-Sparse FlashAttention	✓ 56.0%	✓ 63.1%

Entirely new capability: longer context unlocks tasks no prior Transformer could solve

First Transformer to Solve Path-X

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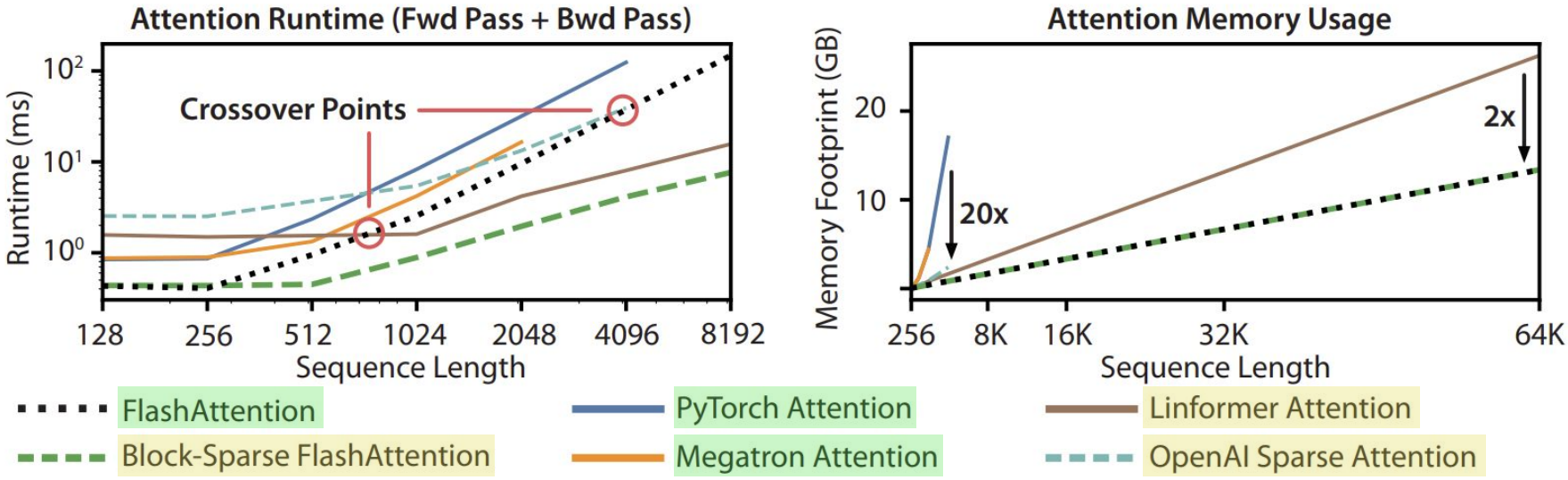
Trains to 64k input tokens, but still fails

Unclear why this is random

FlashAttention performs better than Block-Sparse FlashAttention because it is exact

Training Runtime and Memory

Comparing runtime and memory footprint of multiple implementations across different sequence lengths



Limitations and Next Steps

Limitations

Requires CUDA

Hand-written GPU kernels
Not portable across
architectures

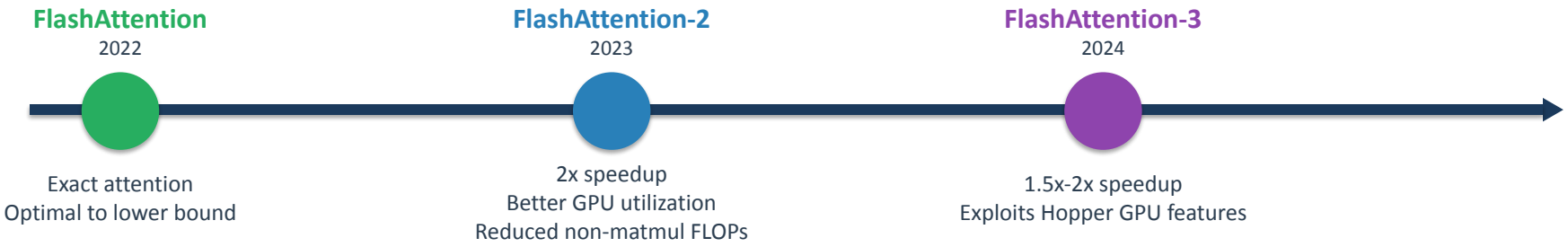
Single-GPU Optimal

Multi-GPU adds another
layer of IO analysis
(inter-GPU transfers)

Head Dim Constraint

Less speedup at d=128
vs d=64 (larger blocks
need more SRAM)

Next Steps and Extensions



FLOPs aren't the whole picture. IO and GPU awareness is equally, if not more, important

Thank You for Listening

Any Questions?